

# Lefschetz classes of simple factors of Fermat Jacobian of prime degree over finite fields

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## Abstract

We give a necessary and sufficient condition in terms of a matrix for which all Tate classes are Lefschetz for simple abelian varieties over an algebraic closure of a finite field. As an application, we prove under an assumption that all Tate classes are Lefschetz for simple factors of Fermat Jacobian of prime degree.

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## 1 Introduction

Let  $p$  be a prime number. Let  $\mathbb{F}_p$  be a finite field with  $p$ -elements and let  $\mathbb{F}$  be a fixed algebraic closure of  $\mathbb{F}_p$ . Let  $\ell$  be a prime number different from  $p$ . Let  $A_0$  be an abelian variety over a finite subfield  $\mathbb{F}_q$  of  $\mathbb{F}$  and let  $A$  be the abelian variety  $A_0 \otimes_{\mathbb{F}_q} \mathbb{F}$  over  $\mathbb{F}$ . There is the cycle map

$$cl_A^i : CH^i(A) \otimes \mathbb{Q}_\ell \longrightarrow H^{2i}(A, \mathbb{Q}_\ell(i)),$$

where  $CH^i(A)$  is the Chow group of algebraic cycles on  $A$  of codimension  $i$  modulo rational equivalence, and  $H^{2i}(A, \mathbb{Q}_\ell(i))$  is the  $\ell$ -adic étale cohomology of  $A$ . Then we know that the image of the map  $cl_A^i$  is contained in the space  $\mathcal{T}_\ell^i(A)$  of  $\ell$ -adic Tate classes of degree  $i$  on  $A$  which is defined as follows:

$$\mathcal{T}_\ell^i(A) := \varinjlim_{L/\mathbb{F}_q} H^{2i}(A, \mathbb{Q}_\ell(i))^{\text{Gal}(\mathbb{F}/L)}.$$

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Here  $L/\mathbb{F}_q$  runs over all finite extensions of  $\mathbb{F}_q$ . We call the elements of the image of  $cl_A^i$  *the algebraic classes* of degree  $i$ .

**Conjecture .** For all  $i \geq 0$ ,  $\text{Im}(cl_A^i) = \mathcal{T}_\ell^i(A)$ .

This is conjectured by Tate [14, Conjecture 1]. In this paper, we call the conjecture the Tate conjecture. The Tate conjecture for  $A$  implies the Tate conjecture for  $A_0/\mathbb{F}_q$ , that is, the cycle map  $cl_{A_0}^i : \text{CH}^i(A_0) \otimes \mathbb{Q}_\ell \longrightarrow H^{2i}(A, \mathbb{Q}_\ell(i))^{\text{Gal}(\mathbb{F}/\mathbb{F}_q)}$  is surjective. On the other hand, Tate [15] proved that  $cl_{A_0}^1$  is surjective for all abelian varieties over finite fields. Therefore all Tate classes of degree 1 are divisor classes on  $A$ . The elements of the  $\mathbb{Q}_\ell$ -subalgebra of  $\mathcal{T}_\ell(A) := \bigoplus_{i \geq 0} \mathcal{T}_\ell^i(A)$  generated by  $\mathcal{T}_\ell^1(A)$  are called the *the Lefschetz classes*

on  $A$ . If all  $\ell$ -adic Tate classes on  $A$  are Lefschetz, then the Tate conjecture holds for  $A$ . But there are examples where not all Tate classes are Lefschetz and the Tate conjecture holds ([9, Example 1.8]). If  $A$  is a product of elliptic curves, then all  $\ell$ -adic Tate classes on  $A$  are Lefschetz by a result of Spiess [13]. Zarhin [19], Lenstra and Zarhin [6] gave other example (c.f. [9, A.7]). Kowalski [5] proved that for certain simple ordinary abelian varieties, all  $\ell$ -adic Tate classes are Lefschetz. In this paper, for a simple factor of the Jacobian of Fermat curve of prime degree, we give a necessary and sufficient condition for which all  $\ell$ -adic Tate classes are Lefschetz.

Let  $m$  be a positive integer prime to  $p$ . Let  $J_m$  be the Jacobian of Fermat curve of degree  $m$  defined by the equation

$$x_0^m + x_1^m + x_2^m = 0 \subset \mathbb{P}_{\mathbb{F}_q}^2.$$

Shioda-Katsura [10, Proposition 3.10] proved that  $J_m$  is isogenous to a product of supersingular elliptic curves if and only if  $p^\nu \equiv -1 \pmod{m}$  for some  $\nu$ . If  $m$  is a prime  $l$  different from  $p$ , then the condition that  $p^\nu \equiv -1 \pmod{l}$  for some  $\nu$  is equivalent to the condition that the residual degree  $f$  of  $p$  in  $\mathbb{Q}(\mu_l)$  is even. Hence if  $f$  is even, then a simple factor  $A$  of  $J_l$  is isogenous to a supersingular elliptic curve over  $\mathbb{F}$ . In this case, all  $\ell$ -adic Tate classes on all powers of  $A$  are Lefschetz. In this paper, we consider in case that  $f$  is odd.

To state main result, we recall Jacobi sum. Let  $\alpha$  be an element of the set

$$\mathcal{A}_l^1 := \{(a_0, a_1, a_2) \mid a_i \in \mathbb{Z}/l, a_i \not\equiv 0, a_0 + a_1 + a_2 \equiv 0\}.$$

Then Jacobi sum  $j(\alpha)$  is defined by

$$j(\alpha) = - \sum_{\substack{1+v_1+v_2=0 \\ v_i \in \mathbb{F}_{p^f}^\times}} \psi(v_1)^{a_1} \psi(v_2)^{a_2}.$$

Here  $\psi$  is a fixed character of order  $l$  of  $\mathbb{F}_{p^f}^\times$ . By definition,  $j(\alpha) \in \mathbb{Q}(\mu_l)$  where  $\mu_l$  is the set of  $l$ -th root of unity. We identify  $(\mathbb{Z}/l)^\times$  with the Galois group of  $\mathbb{Q}(\mu_l)/\mathbb{Q}$ .

Let  $A_0$  be an absolute simple factor of the Jacobian of  $V_l$  over  $\mathbb{F}_{p^f}$  and let  $A := A_0 \otimes \mathbb{F}$ . We denote by  $C(A)$  the center of  $\text{End}_{\mathbb{F}}(A) \otimes \mathbb{Q}$ . Let  $\pi_0$  be the Frobenius endomorphism of  $A_0$ . Then  $\pi_0$  can be given by Jacobi sum (cf. [20]):  $\pi_0 = j(\alpha)$  for some  $\alpha \in \mathcal{A}_l^1$ . Therefore  $C(A) = \mathbb{Q}(j(\alpha))$  and  $C(A) \subset \mathbb{Q}(\mu_l)$ .

Our main result is the following:

**Theorem 1.1.** *Let  $l$  be an odd prime number different from  $p$ . Let  $A$  be a simple factor of the Jacobian of the Fermat curve of degree  $l$  over  $\mathbb{F}$ . Let  $\alpha = (a_0, a_1, a_2)$  be an element of  $\mathcal{A}_l^1$  such that  $C(A) = \mathbb{Q}(j(\alpha))$ . Let  $H_\alpha$  be the Galois group of  $\mathbb{Q}(\mu_l)/C(A)$ . Then all  $\ell$ -adic Tate classes on all powers of  $A$  are Lefschetz if and only if for any odd Dirichlet character  $\chi$  of  $(\mathbb{Z}/l)^\times$  with  $\chi|_{H_\alpha} = 1$ ,  $\sum_{i=0}^2 \chi(a_i) \neq 0$ .*

*In particular, all  $\ell$ -adic Tate classes on all powers of  $A$  are Lefschetz in the following cases:*

- (1)  $[C(A) : \mathbb{Q}] \not\equiv 0 \pmod{6}$ ,
- (2)  $\alpha$  is equal to an element of the form  $(a, a, b)$  up to permutation,
- (3)  $[C(A) : \mathbb{Q}] = 2^{s+1} \cdot 3$  ( $s \geq 0$ ).

**Corollary 1.2** (Corollary 4.1). *Let  $l$  be an odd prime number different from  $p$ . Let  $J(C_l)$  be the Jacobian variety of the hyperelliptic curve*

$$C_l : y^2 = x^l - 1.$$

*Then all  $\ell$ -adic Tate classes on all powers of  $J(C_l)$  are Lefschetz.*

This corollary is an analogue of Shioda's result on Hodge conjecture for  $J(C_l)/\mathbb{C}$  ([12, Corollary 5.3]). In case that  $p \equiv 1 \pmod{l}$ , Shioda's argument in [12] works over finite fields and shows that the above corollary holds. The argument also shows that not all  $\ell$ -adic Tate classes are Lefschetz and the Tate conjecture holds for  $J(C_9)$ . However in case that  $p \not\equiv 1 \pmod{l}$ , the argument needs a similar result to the key lemma in Shioda's argument which is not proven (cf. [11, p. 181, Lemma]). In this paper, we use other argument. More precisely we use Milne's result [9] on the Tate conjecture (see §2).

The key of the proof of Theorem 1.1 is to give a necessary and sufficient condition for which all  $\ell$ -adic Tate classes are Lefschetz for simple abelian varieties in terms of a matrix as follows: for a simple abelian variety  $A$  over

$\mathbb{F}$ , we define a matrix  $T_A$  whose rank depends only on  $A$  (see §3). Using Milne's result, we prove the following:

**Theorem 1.3** (Theorem 3.1). *Let  $A$  be a simple abelian variety over  $\mathbb{F}$  of dimension  $\geq 2$ . Then all  $\ell$ -adic Tate classes on all powers of  $A$  are Lefschetz if and only if  $\text{rank } T_A = r$ . Here  $r$  is the half of the number of all distinct embeddings  $C(A) \rightarrow \mathbb{C}$ .*

This paper is organized as follows: in §2, we recall Milne's result on the Tate conjecture. In §3, we prove Theorem 1.3 and we calculate the matrix  $T_A$  in a special case (Theorem 3.4). In the last section, we prove Theorem 1.1 using the necessary and sufficient condition.

## Notation

Through this paper,  $\mathbb{Q}^{\text{al}}$  denotes the algebraic closure of  $\mathbb{Q}$  in  $\mathbb{C}$ . For a finite étale  $\mathbb{Q}$ -algebra  $E$ ,  $\Sigma_E := \text{Hom}(E, \mathbb{Q}^{\text{al}})$ . If  $E$  is a field Galois over  $\mathbb{Q}$ , we identify  $\Sigma_E$  with the Galois group  $\text{Gal}(E/\mathbb{Q})$ .

For a finite set  $S$ ,  $\mathbb{Z}^S$  denotes the set of functions  $f : S \rightarrow \mathbb{Z}$ .

An affine algebraic group is of multiplicative type if it is commutative and its identity component is a torus. For such a group  $W$  over  $\mathbb{Q}$ ,  $\chi(W) := \text{Hom}(W_{\mathbb{Q}^{\text{al}}}, \mathbb{G}_m)$  denotes the group of characters of  $W$ .

For a finite étale  $\mathbb{Q}$ -algebra  $E$ ,  $(\mathbb{G}_m)_{E/\mathbb{Q}}$  denotes the Weil restriction  $\text{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)$  which is characterized by  $\chi((\mathbb{G}_m)_{E/\mathbb{Q}}) = \mathbb{Z}^{\Sigma_E}$ .

For an abelian variety  $A$  over  $\mathbb{F}$ ,  $\text{End}_{\mathbb{F}}^0(A)$  denotes  $\text{End}_{\mathbb{F}}(A) \otimes \mathbb{Q}$ , and  $C(A)$  denotes the center of  $\text{End}_{\mathbb{F}}^0(A)$ .

## 2 Milne's result on the Tate conjecture

We recall Milne's result on the Tate conjecture. For an abelian variety  $A$  over  $\mathbb{F}$ , there are three important groups of multiplicative type  $L(A)$ ,  $M(A)$  and  $P(A)$  which act on  $H^{2*}(A) := \bigoplus_{i \geq 0} H^{2i}(A, \mathbb{Q}_{\ell}(i))$ . These groups are intro-

duced by Milne [7], [8], [9], who proved that these groups are characterized by the following properties (see [9, p. 14, Lemma]): for all  $r$ ,

- (a)  $H^{2*}(A^r)^{L(A)}$  is the space of Lefschetz classes.
- (b)  $H^{2*}(A^r)^{M(A)}$  is the space of algebraic classes, provided numerical equivalence coincides with  $\ell$ -adic homological equivalence.
- (c)  $H^{2*}(A^r)^{P(A)}$  is the space of  $\ell$ -adic Tate classes on  $A^r$ .

Statement (a) is proved by Milne [7]. Statement (b) is proved by using a result of Jannsen [4]. Clozel [1] proved that for an abelian variety  $A$  over  $\mathbb{F}$ , there is a set of primes  $\ell$  of positive density for which  $\ell$ -adic homological equivalence and numerical equivalence coincide. Deligne [2] gives a more precise version of Clozel's result. Statement (c) is almost by definition of  $P(A)$ . The following theorem is due to Milne [9, p. 14, Theorem]:

**Theorem 2.1.** *Let  $A$  be an abelian variety over  $\mathbb{F}$ . Then  $P(A) \subset M(A) \subset L(A)$ , and*

- (i) *the Tate conjecture holds for  $A$  if and only if  $P(A) = M(A)$ ;*
- (ii) *all  $\ell$ -adic Tate classes on all powers of  $A$  are Lefschetz if and only if  $P(A) = L(A)$ .*

We now recall the definitions of the Lefschetz group and the group  $P$  associated to an abelian variety over  $\mathbb{F}$ , and recall a description of the character group of these groups. For properties and results on the groups, we refer to Milne [7], [8] and [9].

Let  $A$  be an abelian variety over  $\mathbb{F}$ . A polarization  $\lambda : A \rightarrow A^\vee$  of  $A$  determines an involution of  $\text{End}_{\mathbb{F}}^0(A)$  which stabilizes  $C(A)$ . The restriction of the involution to  $C(A)$  is independent of the choice of  $\lambda$ . By  $\dagger$ , we denote this restriction on  $C(A)$ .

**Definition 2.2** ([7, 4.3, 4.4], [8, pp.52-53], [9, A.3]). The *Lefschetz group*  $L(A)$  of  $A$  is the algebraic group over  $\mathbb{Q}$  such that

$$L(A)(R) = \{\alpha \in (C(A) \otimes R)^\times \mid \alpha\alpha^\dagger \in R^\times\}$$

for all  $\mathbb{Q}$ -algebras  $R$ .

We can describe  $L(A)$  as a subgroup of  $(\mathbb{G}_m)_{C(A)/\mathbb{Q}}$  in terms of characters as follows ([9, A.7]):  $L(A)$  is a subgroup of  $(\mathbb{G}_m)_{C(A)/\mathbb{Q}}$  whose character group is

$$\frac{\mathbb{Z}^{\Sigma_{C(A)}}}{\{g \in \mathbb{Z}^{\Sigma_{C(A)}} \mid g = \iota g \text{ and } \sum g(\sigma) = 0\}}. \quad (2.1)$$

Here  $\iota g$  is a function sending an element  $\sigma$  of  $\Sigma_{C(A)}$  to  $g(\iota\sigma)$ , and  $\sum g(\sigma)$  denotes  $\sum_{\sigma \in \Sigma_{C(A)}} g(\sigma)$ .

**Definition 2.3** ([8, §4], [9, A.7]). Let  $A_0$  be a model of  $A$  over a finite field  $\mathbb{F}_q \subset \mathbb{F}$ , and let  $\pi_0$  be the Frobenius of  $A_0$ . Then the group  $P(A)$  of  $A$  is the smallest algebraic subgroup of  $L(A)$  containing some power of  $\pi_0$ . It is independent of the choice of  $A_0$ .

To state Milne's result on the character group of  $P$ , we introduce Weil numbers and some notion which is related to Weil numbers. A Weil  $q$ -number of weight  $i$  is an algebraic number  $\pi$  such that  $q^N \pi$  is an algebraic integer for some  $N$  and the complex absolute value  $|\sigma(\pi)|$  is  $q^{i/2}$ , for all embeddings  $\sigma : \mathbb{Q}[\pi] \rightarrow \mathbb{C}$ . Then  $\pi$  is a unit at all primes of  $\mathbb{Q}[\pi]$  not dividing  $p$ . We define the *slope function*  $s_\pi$  of  $\pi$  as follows: for any prime  $\mathfrak{p}$  dividing  $p$  of a field containing  $\pi$ ,

$$s_\pi(\mathfrak{p}) = \frac{\text{ord}_{\mathfrak{p}}(\pi)}{\text{ord}_{\mathfrak{p}}(q)}. \quad (2.2)$$

For the definition of Weil numbers,  $s_\pi(\mathfrak{p}) + s_\pi(\iota\mathfrak{p}) = i (= wt(\pi))$ . Here  $\iota$  is complex conjugation on  $\mathbb{C}$ . The slope function determines a Weil  $q$ -number up to a root of unity.

Let  $\pi$  be a Weil  $p^f$ -number and let  $\pi'$  be a Weil  $p^{f'}$ -number. We say  $\pi$  and  $\pi'$  are *equivalent* if  $\pi^{f'} = \pi'^f \cdot \zeta$  for some root of unity  $\zeta$ . We define a *Weil germ* to be an equivalent class of Weil numbers. For a Weil germ  $\pi$ , the slope function and weight of  $\pi$  are the slope function (see (2.2)) and weight of any representative of  $\pi$ , and  $\mathbb{Q}[\pi]$  is defined to be the smallest subfield of  $\mathbb{Q}^{\text{al}}$  containing a representative of  $\pi$ .

Now assume that  $A$  is simple and that  $\text{End}_{\mathbb{F}}^0(A) = \text{End}_{\mathbb{F}_q}^0(A_0)$ . Then the Frobenius endomorphism  $\pi_0$  of  $A_0$  generates  $C(A)$  over  $\mathbb{Q}$ , i.e.  $C(A) = \mathbb{Q}[\pi_0]$ . We know that the Frobenius endomorphism  $\pi_0$  of  $A_0$  is a Weil  $q$ -number of weight 1. Let  $\pi_A$  denote the germ represented by  $\pi_0$ . Milne's result on the character of  $P(A)$  is the following ([9, A.7]): let  $g$  be a character of  $L(A)$ . Then  $g$  is trivial on  $P(A)$  if and only if for all primes  $v$  dividing  $p$  of a field containing all conjugates  $\sigma(\pi_0)$ ,

$$\sum_{\sigma \in \Sigma_{C(A)}} g(\sigma) s_{\sigma\pi_A}(v) = 0. \quad (2.3)$$

Note that  $s_{\sigma\pi_A}(v) = s_{\pi_A}(\sigma^{-1}v)$ .

### 3 Necessary and sufficient condition

Let  $A$  be a simple abelian variety over  $\mathbb{F}$ . Let  $A_0$  be a model of  $A$  over a finite subfield  $\mathbb{F}_q \subset \mathbb{F}$  with property that  $\text{End}_{\mathbb{F}_q}(A_0) = \text{End}_{\mathbb{F}}(A)$ . We take a finite Galois extension  $K$  of  $\mathbb{Q}$  containing all conjugates of  $C(A)$ . Let  $\{\mathfrak{p}_1, \dots, \mathfrak{p}_d\}$  be the set of primes of  $K$  dividing  $p$ . We assume that  $\dim A \geq 2$ . Then  $C(A)$  is totally imaginary and CM field (cf. [16, p.97]).

Let  $\{\sigma_1, \dots, \sigma_r, \iota\sigma_1, \dots, \iota\sigma_r\}$  be the set of all distinct embeddings  $C(A) \rightarrow \mathbb{C}$ . We define a  $d \times r$  matrix  $T_A$  as follows:

$$T_A = (a_{ij}), \quad a_{ij} := s_{\sigma_j \pi_0}(\mathfrak{p}_i) - \frac{1}{2} \quad (1 \leq i \leq d, 1 \leq j \leq r).$$

Then the matrix  $T_A$  is independent of the choice of the model  $A_0/\mathbb{F}_q$ , but depends on the ordering of the  $\mathfrak{p}_i$  and the  $\sigma_j$ . The rank of  $T_A$  depends only on  $A$ . Using Milne's result on the character group of  $L(A)$  and  $P(A)$ , we prove the following:

**Theorem 3.1.** *Let  $A$  be a simple abelian variety over  $\mathbb{F}$  of dimension  $\geq 2$ . Then  $P(A) = L(A)$  if and only if  $\text{rank } T_A = r$ .*

We first describe the kernel of the natural map  $\mathbb{Z}^{\Sigma_{C(A)}} \rightarrow \chi(L(A))$  in terms of a matrix. Let  $J$  be the  $(r+1) \times 2r$  matrix

$$\begin{pmatrix} I_r & -I_r \\ 0 & B \end{pmatrix}, \quad B = (1 \quad \dots \quad 1)$$

where  $I_r$  is the  $r \times r$  identity matrix. We consider the set of character functions for  $\sigma_i, \iota\sigma_i$  as a basis of  $\mathbb{Z}^{\Sigma_{C(A)}}$ . We then have the following:

**Lemma 3.2.** *The kernel of the natural map  $\phi : \mathbb{Z}^{\Sigma_{C(A)}} \rightarrow \chi(L(A))$  coincides with the kernel of the map  $\mathbb{Z}^{\Sigma_{C(A)}} \rightarrow \mathbb{Z}^{r+1}$  defined by  $J$ .*

*Proof.* From (2.1), the kernel of  $\phi$  is  $\{g \in \mathbb{Z}^{\Sigma_{C(A)}} \mid g = \iota g \text{ and } \sum g(\sigma) = 0\}$ . Here  $\iota g$  is a function sending an element  $\sigma$  of  $\Sigma_{C(A)}$  to  $g(\iota\sigma)$ , and  $\sum g(\sigma)$  denotes  $\sum_{\sigma \in \Sigma_{C(A)}} g(\sigma)$ . Therefore the kernel of  $\phi$  is the kernel of the map  $\mathbb{Z}^{\Sigma_{C(A)}} \rightarrow \mathbb{Z}^{2r+1}$  defined by the  $(2r+1) \times 2r$  matrix

$$J' := \begin{pmatrix} I_r & -I_r \\ -I_r & I_r \\ B & B \end{pmatrix}.$$

By row operations,  $J'$  is equivalent over  $\mathbb{Z}$  to

$$\begin{pmatrix} I_r & -I_r \\ 0 & 2B \\ 0_r & 0_r \end{pmatrix}$$

where  $0_r$  is the  $r \times r$  matrix with all entries equal to 0. From this we have  $\text{Ker}(J) = \text{Ker}(J')$ .  $\square$

Next we describe the condition (2.3) in terms of a matrix. Let  $T'$  be the  $d \times 2r$  matrix  $(U \ V)$ , where  $U$  is the  $d \times r$  matrix  $(s_{\sigma_j \pi_0}(\mathbf{p}_i))$  and  $V$  is the  $d \times r$  matrix  $(s_{\iota \sigma_j \pi_0}(\mathbf{p}_i))$  ( $1 \leq i \leq d, 1 \leq j \leq r$ ). From (2.3), a character  $g$  of  $L(A)$  is trivial on  $P(A)$  if and only if  $g$  belongs to the kernel of the map  $\mathbb{Z}^{\Sigma_{C(A)}} \rightarrow \mathbb{Z}^d$  defined by  $T'$ . By Lemma 3.2,  $L(A) = P(A)$  if and only if  $\text{Ker}(J) = \text{Ker}(T')$ . Hence we deduce Theorem 3.1 from the following lemma:

**Lemma 3.3.** *Let notation be as above.*

- (1)  $\text{rank } T' = \text{rank } T_A + 1$ .
- (2)  $\text{Ker}(J) = \text{Ker}(T')$  if and only if  $\text{rank } T' = r + 1$

*Proof.* (1) From the equation  $s_{\sigma_j \pi_0}(\mathbf{p}_i) + s_{\iota \sigma_j \pi_0}(\mathbf{p}_i) = 1$  for all  $i, j$ , we easily see that the matrix  $T'$  is column equivalent to the  $d \times 2r$  matrix

$$(C \ 0_{d \times (r-1)}), \quad C = \begin{pmatrix} & 1 \\ U & \vdots \\ & 1 \end{pmatrix}$$

where  $0_{d \times (r-1)}$  is the  $d \times (r-1)$  matrix with all entries equal to 0.

Since the complex conjugation  $\iota \in G$  acts on the set  $\{\mathbf{p}_1, \dots, \mathbf{p}_d\}$ , after renumbering the  $\mathbf{p}_i$  if necessary, there is a positive integer  $t$  such that

$$\begin{aligned} \iota \mathbf{p}_i &= \mathbf{p}_{i+t} & \text{for } 1 \leq i \leq t, \\ \iota \mathbf{p}_i &= \mathbf{p}_i & \text{for } 2t+1 \leq i \leq d. \end{aligned}$$

From this, we obtain that for each  $j$ ,

$$\begin{aligned} s_{\sigma_j \pi_0}(\mathbf{p}_i) + s_{\sigma_j \pi_0}(\mathbf{p}_{i+t}) &= 1 & \text{for } 1 \leq i \leq t, \\ s_{\sigma_j \pi_0}(\mathbf{p}_i) &= \frac{1}{2} & \text{for } 2t+1 \leq i \leq d. \end{aligned} \tag{3.1}$$

These equations show that the matrix  $C$  is row equivalent to the matrix

$$\begin{pmatrix} U' & 0_{(t+1) \times (r-1)} \\ 0_{(d-t+1) \times (r+1)} & 0_{(d-t+1) \times (r-1)} \end{pmatrix}, \quad U' = \begin{pmatrix} & 0 \\ U'' & \vdots \\ & 0 \\ 1 & \dots & 1 & 2 \end{pmatrix}$$

where  $U''$  is the  $t \times r$  matrix  $(s_{\sigma_j \pi_0}(\mathbf{p}_i) - \frac{1}{2})$  ( $1 \leq i \leq t, 1 \leq j \leq r$ ).

On the other hand, from (3.1) we also obtain that the matrix  $T_A$  is row equivalent to the matrix

$$\begin{pmatrix} U'' \\ 0_{(d-t) \times r} \end{pmatrix}.$$



Hence

$$\text{rank } T' = \text{rank } C = \text{rank } U'' + 1 = \text{rank } T_A + 1.$$

(2) If  $\text{Ker}(J) = \text{Ker}(T')$ , then clearly  $\text{rank } T' = r + 1$ .

Conversely assume that  $\text{rank } T' = r + 1$ . Then  $T'$  is row equivalent to a matrix of the following form

$$\begin{pmatrix} I_{r+1} & D \\ 0_{(d-r-1) \times (r+1)} & 0_{(d-r-1) \times (r-1)} \end{pmatrix}.$$

Put  $D = (b_{ij})$ . From the equation  $s_{\sigma_j \pi_0}(\mathfrak{p}_i) + s_{\iota \sigma_j \pi_0}(\mathfrak{p}_i) = 1$ , we have

$$\begin{aligned} 1 + b_{ii} &= 1 \quad \text{if } i \neq 1, r+1, \\ 0 + b_{ij} &= 1 \quad \text{otherwise.} \end{aligned}$$

Hence the matrix  $(I_{r+1} \ D)$  is row equivalent to the matrix  $J$ , which implies  $\text{Ker}(T') = \text{Ker}((I_{r+1} \ D)) = \text{Ker}(J)$ .  $\square$

### 3.1 Calculation in special case

Applying Theorem 2.1 and Theorem 3.1, we have the following:

**Theorem 3.4.** *Let  $A$  be a simple abelian variety over  $\mathbb{F}$  of dimension  $\geq 2$ . Assume that  $C(A)/\mathbb{Q}$  is abelian with Galois group  $G$ . Then all  $\ell$ -adic Tate classes on all powers of  $A$  are Lefschetz if and only if for any character  $\varphi$  of  $G$  with  $\varphi(\iota) = -1$ ,*

$$\sum_{\sigma \in G} e(\sigma) \varphi(\sigma) \neq 0.$$

Here  $e(\sigma) = s_{\pi_0}(\sigma \mathfrak{p}) - \frac{1}{2}$  where  $\mathfrak{p}$  is a prime of  $C(A)$  dividing  $p$ .

In particular, all  $\ell$ -adic Tate classes on all powers of  $A$  are Lefschetz if one of the following condition holds:

- (1)  $G$  is cyclic of order  $2^{s+1}$  ( $s \geq 0$ ),
- (2)  $G$  is cyclic of order  $2^{s+1}l$  with  $l$  odd prime and the order of  $\text{End}_{\mathbb{F}}^0(A)$  in the Brauer group  $\text{Br}(C(A))$  of  $C(A)$  is odd.

To prove this theorem, we need the following proposition:

**Proposition 3.5.** *Let  $A$  be a simple abelian variety over  $\mathbb{F}$ . Assume that  $C(A)$  is Galois over  $\mathbb{Q}$ . Let  $G$  be the Galois group of  $C(A)$  over  $\mathbb{Q}$ . Let  $\mathfrak{p}$  be a prime ideal of  $C(A)$  lying over  $p$ . If the decomposition group  $G_{\mathfrak{p}}$  of  $\mathfrak{p}$  in  $C(A)$  is a normal subgroup of  $G$ , then  $G_{\mathfrak{p}} = 1$ , namely  $p$  is completely decomposed in  $C(A)$ .*

*Proof.* Let a prime decomposition of  $\pi_0$  in  $C(A)$  be as follows:

$$(\pi_0) = \prod_{\sigma \in G/G_p} \sigma(\mathfrak{p})^{e_\sigma}.$$

Let  $\tau$  be an element of  $G_p$ . Since  $G_p$  is normal, we have  $(\pi_0) = (\tau\pi_0)$  as ideals. From Lemma 3.6 below, we obtain that  $\tau = 1$  and that  $G_p = 1$ .  $\square$

**Lemma 3.6.** *Let  $\pi$  be a Weil germ. Let  $\pi_0 \in \mathbb{Q}[\pi]$  be a representative of  $\pi$ . Let  $K$  be a Galois extension of  $\mathbb{Q}$  such that  $\mathbb{Q}[\pi] \subset K$ . Then there is no elements  $\sigma \in \text{Gal}(K/\mathbb{Q})$  satisfying the following conditions:*

- (1)  $\sigma$  fix the ideal  $(\pi_0)$ ,
- (2)  $\sigma$  is not trivial on  $\mathbb{Q}[\pi]$ .

*Proof.* We assume that there is an element  $\sigma \in \text{Gal}(K/\mathbb{Q})$  satisfying conditions (1) and (2). From condition (1), there is an unit  $u$  of the integer ring of  $K$  such that  $\pi_0 = u \cdot \sigma\pi_0$ . For any  $\tau \in \text{Gal}(K/\mathbb{Q})$ , we have  $|\tau u| = 1$  since  $|\tau\pi_0| = q^{1/2}$ . Here we used that  $\pi_0$  is a Weil  $q$ -number. Hence  $u$  is a root of unity. Therefore we have  $\pi_0^m = \sigma\pi_0^m$  for some  $m > 0$ . Since  $\sigma$  acts on the subfield  $\mathbb{Q}(\pi_0^m)$  of  $\mathbb{Q}[\pi]$  trivially,  $\mathbb{Q}(\pi_0^m)$  is not equal to  $\mathbb{Q}[\pi]$  by condition (2).

On the other hand, since  $\mathbb{Q}[\pi]$  is the smallest field containing a representative of  $\pi$ , we obtain that  $\mathbb{Q}[\pi] \subset \mathbb{Q}(\pi_0^m)$ . Hence  $\mathbb{Q}(\pi_0^m) = \mathbb{Q}[\pi]$  which is a contradiction.  $\square$

*Proof of Theorem 3.4.* By Theorem 2.1 and Theorem 3.1, our task is reduced to calculate the matrix  $T_A$ . Since  $C(A)$  is Galois over  $\mathbb{Q}$ , we may take  $K = C(A)$ . Put  $G = \{\sigma_1, \dots, \sigma_r, \iota\sigma_1, \dots, \iota\sigma_r\}$ . Here  $\iota \in G$  is the complex conjugate on  $C(A)$ . Let  $\mathfrak{p}$  be a prime of  $C(A)$  dividing  $p$ . By Proposition 3.5, the set  $\{\sigma\mathfrak{p} \mid \sigma \in G\}$  is the set of all primes of  $C(A)$  dividing  $p$ . Let  $e$  be the function  $G \rightarrow \mathbb{C}$  defined as  $e(\sigma) = s_{\pi_0}(\sigma\mathfrak{p}) - \frac{1}{2}$  for  $\sigma \in G$ . From the proof of Theorem 3.1, the matrix  $T_A$  is row equivalent to the matrix

$$\begin{pmatrix} U'' \\ 0_{r \times r} \end{pmatrix}$$

where  $U''$  is the  $r \times r$  matrix  $(e(\sigma_i\sigma_j^{-1}))$  ( $1 \leq i, j \leq r$ ). Since  $\text{rank } T_A = \text{rank } U''$ , we obtain that  $\text{rank } T_A = r$  if and only if  $\det(U'') \neq 0$ .

Now we calculate  $\det(U'')$ . Let  $\phi$  be the fixed character of  $G$  with  $\phi(\iota) = -1$ . Let  $f : G \rightarrow \mathbb{C}$  be the function defined as  $f(\sigma) = \phi(\sigma)e(\sigma)$  for  $\sigma \in G$ . Then we have

$$\det(U'') = \det\left((f(\sigma_i\sigma_j^{-1}))\right).$$

For any  $\sigma \in G$ , we have

$$f(\iota\sigma) = \phi(\iota\sigma)e(\iota\sigma) = -\phi(\sigma)(-e(\sigma)) = f(\sigma).$$

Hence  $f$  is a function on  $G' := G/\{1, \iota\}$ . Now we need the following lemma:

**Lemma 3.7.** *Let  $G$  be a finite abelian group and let  $f$  be a function on  $G$  with values in some field of characteristic 0. Then*

$$\det(f(\sigma\tau^{-1}))_{\sigma, \tau \in G} = \prod_{\psi} \sum_{\sigma \in G} f(\sigma)\psi(\sigma).$$

Here  $\psi$  runs over all character of  $G$ .

For the proof of this lemma, see [18, Lemma 5.26].

From this lemma, we have

$$\det(f(\sigma\tau^{-1}))_{\sigma, \tau \in G'} = \prod_{\psi} \sum_{\sigma \in G'} f(\sigma)\psi(\sigma). \quad (3.2)$$

Here  $\psi$  runs over all character of  $G'$ . Furthermore, by elementary calculation, we have

$$\text{RHS of (3.2)} = \frac{1}{2} \prod_{\varphi} \sum_{\sigma \in G} e(\sigma)\varphi(\sigma).$$

Here  $\varphi$  runs over all character of  $G$  with  $\varphi(\iota) = -1$ . By the above argument,  $\text{rank } T_A = r$  if and only if for any such  $\varphi$ ,

$$\sum_{\sigma \in G} e(\sigma)\varphi(\sigma) \neq 0.$$

This completes the proof of the first assertion of Theorem 3.4.

Now we put  $E(\varphi) := \sum_{\sigma \in G} e(\sigma)\varphi(\sigma)$  and prove that  $E(\varphi) \neq 0$  in some cases. Let  $g$  be a generator of  $G$ . In case that condition (1) holds, then  $E(\varphi) = 2 \sum_{i=0}^{2^s-1} e(g^i)\varphi(g^i)$ . Since  $\varphi(g)$  is a primitive  $2^{s+1}$ -th root of unity, we have  $[\mathbb{Q}(\zeta) : \mathbb{Q}] = 2^s$ . Therefore the set  $\{1, \zeta, \zeta^2, \dots, \zeta^{2^s-1}\}$  is a base of  $\mathbb{Q}(\zeta)$  over  $\mathbb{Q}$ . Therefore if  $E(\varphi) = 0$ , then we have

$$e(1) = e(g) = \dots = e(g^{2^s-1}) = 0.$$

From this, we have  $(\pi_0) = (p^n)$  as ideal in  $C(A)$  which is fixed by the action of  $\iota \in G$ . Here  $n := \text{ord}_{\mathfrak{p}}(\pi_0)$ . By Lemma 3.6, we have  $\iota = 1$ , which is a contradiction. Therefore  $E(\varphi) \neq 0$ .

Next we consider in case that the order of  $G$  is  $2^{s+1}l$  with  $l$  an odd prime. Then  $\varphi(g)$  is a primitive  $2^{s+1}$ -th root of unity or a primitive  $2^{s+1}l$ -th root of unity. If  $\varphi(g)$  is a primitive  $2^{s+1}l$ -th root of unity and if  $E(\varphi) = 0$ , then we have

$$\sum_{i=0}^{r2^s l - 1} e(g^i) x^i = h(x) \cdot (b_1 x^{2^s - 1} + b_2 x^{2^s - 2} + \cdots + b_{2^s - 1} x + b_{2^s})$$

for some  $b_i \in \mathbb{Q}$ . Here  $h(x) := x^{2^s(l-1)} - x^{2^s(l-2)} + \cdots - x^{2^s} + 1$  is the minimal polynomial to  $\varphi(g)$ . From this equation, we obtain that for  $0 \leq t \leq 2^s - 1$  and for  $0 \leq i \leq l - 1$ ,

$$e(g^{2^s i + t}) = (-1)^i e(g^t). \quad (3.3)$$

Then the ideal  $(\pi_0)$  is fixed by  $g^l$  because equation (3.3) holds. This is a contradiction by Lemma 3.6. Hence  $E(\varphi) \neq 0$ .

If  $\varphi(g)$  is a primitive  $2^{s+1}$ -th root of unity and if  $E(\varphi) = 0$ , then by a similar argument in case that  $\varphi(g)$  is a primitive  $2^{s+1}l$ -th root of unity, for each  $0 \leq t \leq 2^s - 1$  we have

$$\sum_{i=0}^{l-1} (-1)^i e(g^{2^s i + t}) = 0.$$

On the other hand, let  $n$  be the order of  $\text{End}_{\mathbb{F}}^0(A)$  in the Brauer group  $\text{Br}(C(A))$  of  $C(A)$ . By class field theory,  $n$  is the smallest integer such that  $n \cdot \text{inv}_v(\text{End}_{\mathbb{F}}^0(A))$  belongs to  $\mathbb{Z}$  for all prime  $v$  of  $C(A)$ . We here have

$$\text{inv}_v(\text{End}_{\mathbb{F}}^0(A)) = s_\pi(v)[C(A)_v : \mathbb{Q}_p].$$

By Proposition 3.5,  $[C(A)_v : \mathbb{Q}_p] = 1$ . Hence, for any  $1 \leq t \leq 2^s$  we have

$$n \cdot \sum_{i=0}^{l-1} (-1)^i e(g^{2^s i + t}) \equiv \frac{n}{2} \pmod{\mathbb{Z}}.$$

Since  $n$  is odd from the assumption,  $\sum_{i=0}^{l-1} (-1)^i e(g^{2^s i + t}) \neq 0$ . Therefore  $E(\varphi) \neq 0$ . □

## 4 Proof of Theorem 1.1

We here prove Theorem 1.1. We introduce some notation: For any  $c \in (\mathbb{Z}/l)^\times$ , we denote by  $\langle c \rangle$  the least natural number such that  $\langle c \rangle \equiv c \pmod{l}$ . We write  $H$  for the subgroup of  $(\mathbb{Z}/l)^\times$  generated by  $p$ . We identify  $(\mathbb{Z}/l)^\times$  with the Galois group of  $\mathbb{Q}(\mu_l)/\mathbb{Q}$ .

*Proof of Theorem 1.1.* By a result of Shioda-Katsura mentioned in Introduction, we may assume that the residual degree  $f$  of  $p$  in  $\mathbb{Q}(\mu_l)$  is odd and that  $\dim A \geq 2$ . By González's result [3, Theorem 3.3], we have

$$H_\alpha = \left\{ c \in (\mathbb{Z}/l)^\times \mid \sum_{h \in H} \sum_{i=0}^2 \langle hca_i \rangle = \sum_{h \in H} \sum_{i=0}^2 \langle ha_i \rangle \right\}. \quad (4.1)$$

Let  $G$  be the Galois group of  $C(A)/\mathbb{Q}$ . Then we have

$$G \simeq (\mathbb{Z}/l)^\times / H_\alpha.$$

Let  $\iota \in G$  be the complex conjugation on  $C(A)$ . Then to give a character  $\varphi$  of  $G$  with  $\varphi(\iota) = -1$  is equivalent to give a odd Dirichlet character  $\chi$  of  $(\mathbb{Z}/l)^\times$  with  $\chi|_{H_\alpha} = 1$ . By Theorem 3.4, it suffices to show that for any odd Dirichlet character  $\chi$  of  $(\mathbb{Z}/l)^\times$  with  $\chi|_{H_\alpha} = 1$ ,  $\sum_{\sigma \in G} e(\sigma)\chi(\sigma) = 0$  if and only

if  $\sum_{i=0}^2 \chi(a_i) = 0$ . Here  $e(\sigma) = s_{j(\alpha)}(\sigma\mathfrak{p}) - \frac{1}{2}$  where  $\mathfrak{p}$  is a fixed prime of  $C(A)$  dividing  $p$ .

Therefore for a such Dirichlet character  $\chi$ , we calculate  $\sum_{\sigma \in G} e(\sigma)\chi(\sigma)$ . We first consider  $e(\sigma)$ . Let  $\mathfrak{q}$  be a prime of  $\mathbb{Q}(\mu_l)$  dividing  $\mathfrak{p}$ . Then we have

$$e(\sigma) = s_{j(\alpha)}(c\mathfrak{q}) - \frac{1}{2}.$$

Here  $c \in (\mathbb{Z}/l)^\times$  is a representative of  $\sigma$ . Hence we have

$$\sum_{\sigma \in G} e(\sigma)\chi(\sigma) = \frac{1}{d} \sum_{c \in (\mathbb{Z}/l)^\times} e(c)\chi(c), \quad (4.2)$$

where  $d$  is the cardinality of  $H_\alpha$  and  $e(c) = s_{j(\alpha)}(c\mathfrak{q}) - \frac{1}{2}$ . By the ideal decomposition of  $j(\alpha)$  in  $\mathbb{Q}(\mu_l)$  (cf. [10]), we have

$$\begin{aligned} s_{j(\alpha)}(c\mathfrak{q}) - \frac{1}{2} &= \frac{1}{f} \sum_{h \in H} \sum_{i=0}^2 \left( \frac{\langle ha_i c^{-1} \rangle}{l} - 1 \right) - \frac{1}{2} \\ &= \frac{1}{fl} \sum_{i=0}^2 \sum_{h \in H} \left( \langle ha_i c^{-1} \rangle - \frac{l}{2} \right). \end{aligned}$$

From this equation and (4.2),

$$\begin{aligned}
\frac{1}{d} \sum_{c \in (\mathbb{Z}/l)^\times} e(c) \chi(c) &= \frac{1}{fld} \sum_{c \in (\mathbb{Z}/l)^\times} \sum_{i=0}^2 \sum_{h \in H} \left( \langle ha_i c^{-1} \rangle - \frac{l}{2} \right) \chi(c) \\
&= \frac{1}{fld} \sum_{i=0}^2 \sum_{h \in H} \sum_{c \in (\mathbb{Z}/l)^\times} \left( \langle ha_i c^{-1} \rangle - \frac{l}{2} \right) \chi(c) \\
&= \frac{1}{fld} \sum_{i=0}^2 \sum_{h \in H} \chi(a_i)^{-1} \sum_{c \in (\mathbb{Z}/l)^\times} \left( \langle c \rangle - \frac{l}{2} \right) \chi(c)^{-1} \\
&= \frac{1}{ld} \sum_{i=0}^2 \chi(a_i)^{-1} \sum_{c \in (\mathbb{Z}/l)^\times} \langle c \rangle \chi(c)^{-1}.
\end{aligned}$$

Now our task is reduced to show that  $\sum_{c \in (\mathbb{Z}/l)^\times} \langle c \rangle \chi(c)^{-1} \neq 0$ . Let  $L(s, \chi)$  be the  $L$ -series attached to  $\chi$ . Then  $L(1, \chi) \neq 0$  by [18, Corollary 4.4]. Furthermore by [18, Theorem 4.9], we have

$$\sum_{c \in (\mathbb{Z}/l)^\times} \langle c \rangle \chi(c)^{-1} = b \cdot L(1, \chi), \quad (b \in \mathbb{C}^\times).$$

Hence the proof of the first assertion of the theorem is complete.

Next we consider in case (1). Let  $g$  be a generator of  $(\mathbb{Z}/l)^\times$ . Let  $\chi$  be an odd Dirichlet character of  $(\mathbb{Z}/l)^\times$ . Then  $\text{Ker}(\chi)$  is a subgroup of  $(\mathbb{Z}/l)^\times$  generated by  $g^m$  for some  $m \mid (l-1)$ . Now  $l-1 = mk$  for some  $k > 0$ . If  $\sum_{i=0}^2 \chi(a_i) = 0$ , then  $1 + \chi(a_1 a_0^{-1}) + \chi(a_2 a_0^{-1}) = 0$ . Since  $\chi(c)$  is a root of unity, by elementary computation  $\chi(a_1 a_0^{-1})$  is a primitive cubic root of unity. Hence  $m$  is divided by 3. Moreover since  $\chi(-1) = -1$  and  $-1 \equiv g^{\frac{l-1}{2}}$ , we obtain that  $m$  does not divide  $\frac{l-1}{2}$ . Hence  $m$  is even and  $k$  is odd. Therefore  $m \equiv 0 \pmod{6}$ .

From the above argument, we see that if  $l-1 \not\equiv 0 \pmod{6}$ , then the order of  $\chi$  is not divided by 6. Therefore we may assume that  $l-1 \equiv 0 \pmod{6}$ . Let  $H'$  be the subgroup of  $(\mathbb{Z}/l)^\times$  generated by  $g^6$ . If  $[C(A) : \mathbb{Q}] \not\equiv 0 \pmod{6}$ , then  $H_\alpha \not\subset H'$ . For any odd character  $\chi$  of  $(\mathbb{Z}/l)^\times$  with  $\chi|_{H_\alpha} = 1$ , we obtain that  $\text{Ker}(\chi) \not\subset H'$ . Therefore the order of  $\chi$  is not divided by 6.

From the above argument, we see that  $\sum_{i=0}^2 \chi(a_i) \neq 0$  for any such  $\chi$ . Hence the assertion follows from the first assertion of the theorem.

Case (2) is easy. Let  $\chi$  be an odd Dirichlet character of  $(\mathbb{Z}/l)^\times$ . If  $\alpha = (a_0, a_1, a_2) = (a, a, b)$  up to permutation and if  $\sum_{i=0}^2 \chi(a_i) = 0$ , then  $\chi(a^{-1}b) = -2$ . This is a contradiction because  $\phi(c)$  is a root of unity.

Lastly we consider in case (3). From Theorem 3.4(2), it suffices to show that the order  $n$  of  $\text{End}_{\mathbb{F}}^0(A)$  in the Brauer group  $\text{Br}(C(A))$  of  $C(A)$  is odd. By the proof of Theorem 3.4, the order  $n$  is the smallest integer  $n$  such that  $n \cdot s_{j(\alpha)}(v)$  belongs to  $\mathbb{Z}$  for all prime  $v$  of  $C(A)$ . Now  $s_{j(\alpha)}(v) = \frac{m}{f}$  for some  $m \in \mathbb{Z}$ . Therefore  $n$  is a divisor of  $f$ . Since  $f$  is odd,  $n$  is also odd.  $\square$

**Corollary 4.1** (Corollary 1.2). *Let  $l$  be an odd prime number different from  $p$ . Let  $J(C_l)$  be the Jacobian variety of the hyperelliptic curve*

$$C_l : y^2 = x^l - 1.$$

*Then all  $\ell$ -adic Tate classes on all powers of  $J(C_l)$  are Lefschetz.*

*Proof.* The hyperelliptic curve  $C_l$  is a quotient of the Fermat curve  $V_l$  (cf. [12, §5]). By González's result [3, Theorem 3.3],  $J(C_l)$  is a power of a simple abelian variety  $A$  such that  $C(A) = \mathbb{Q}(j(\alpha))$  for some  $\alpha = (a, a, b)$  (cf. [12]). Now the assertion follows from Theorem 1.1(2).  $\square$

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